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Remarks on duality transformations in lattice spin systems†

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Abstract. This contribution deals with the τ-continuum limit of anisotropic plane Ising lattice and in arriving at the quantum dual Hamiltonian by a route different to that of Fradkin and Susskind, one sees the necessity of effecting the exchange of coupling strengths in duality transformations. Construction prescribed by Savit's procedure to arrive at classical dual is also seen to imply this. Implication of this exchange feature is also discussed.

1. Introduction

The role of duality transformations in lattice spin and gauge systems has been studied intensely (Savit 1980, 1982) in recent years leading to new insights into the structure of these transformations. Confining ourselves to the two-dimensional anisotropic Ising square lattice it is known implicitly in the work of Onsager (1944) and Syozi (1972) that the duality transformations necessitate the exchange of coupling strengths in two directions. This exchange feature is more explicitly brought out in later sections by utilising the τ-continuum limit of the lattice system initiated by Fradkin and Susskind (1978) and Kogut (1979). Fradkin and Susskind (1978) and Kogut (1979) consider the τ-continuum limit of the two-dimensional anisotropic Ising lattice and then construct its dual. Instead, in this paper we first construct the classical dual and go to its τ-continuum limit. It is natural to expect that this limit should coincide with the dual of the τ limit obtained earlier. This procedure forcefully brings out the necessity of the exchange of the coupling strengths mentioned earlier. The two procedures are illustrated diagramatically in figure 1. Following the notation of Kogut (1979) the side AB represents the τ-limit procedure and BC going to its quantum dual. The broken lines AD and DC represent the method adopted in this paper.

2. τ-continuum limit

In Fradkin and Susskind (1978) and Kogut (1979) we start from the Hamiltonian of classical lattice models in d dimensions which is looked upon as the action of the system with $(d-1)$ space, the time axis being the other direction. From this viewpoint, the well known transfer matrix along the time axis is then related to the Hamiltonian of the quantum system by the τ-continuum limit procedure. When this is carried out we arrive at the quantum Hamiltonian in terms of non-commuting operators. Thus

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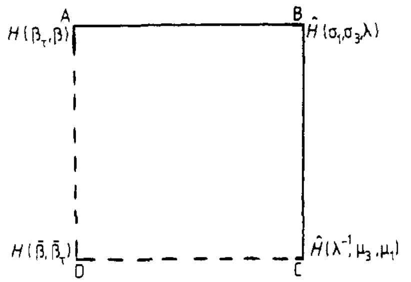

Figure 1.

we end up with a $(d-1)$-dimensional quantum Hamiltonian and aspects of duality have been studied using this method.

To summarise briefly the above, we begin by writing the Hamiltonian of the classical two-dimensional anisotropic Ising system with the Hamiltonian

$$H=\sum_{n_0} L(n_0+1, n_0) \tag{1}$$

and

$$L=\tfrac{1}{2}\beta_\tau \sum_m [S_3(m)-\sigma_3(m)]^2-\tfrac{1}{2}\beta\left[\sum_m (\sigma_3(m+1)\sigma_3(m)+S_3(m+1)S_3(m))\right]. \tag{2}$$

In the above, σ_3 are the spins in a row and $S_3(m)$ spins in the next row β_τ and β are the temporal and spatial coupling strengths. To go to the τ-continuum limit, we have to adjust the coupling strengths β and β_τ as functions of τ such that

$$\beta=\lambda\tau \qquad \text{where} \qquad \lambda=\exp(-2\beta_\tau). \tag{3}$$

To preserve the physics relating to the phase transitions involved here, the temporal couplings have to become very large and the spatial couplings extremely weak. The parameter λ can be related to the scaling relations. Expressing the transfer matrix in this limit as $T=I-\tau\hat{H}$, the Hamiltonian can be identified as

$$\hat{H}=-\sum_m \sigma_1(m)-\lambda\sum_m \sigma_3(m+1)\sigma_3(m). \tag{4}$$

The Hamiltonian in (4) can be interpreted as a one-dimensional Ising system in a transverse magnetic field with $1/\lambda$ representing temperature, the critical point occurring at the value $\lambda=1$. To derive the duality mapping, one defines the operators on the dual lattice as

$$\begin{aligned} \mu_1(n)&=\sigma_3(n+1)\sigma_3(n)\\ \mu_3(n)&=\prod_{m\leqslant n}\sigma_1(m). \end{aligned} \tag{5}$$

Hence the dual Hamiltonian is

$$\hat{H}=-\sum_n \mu_3(n)\mu_3(n+1)-\lambda\sum_n \mu_1(n) \tag{6}$$

i.e.

$$\hat{H}(\sigma;\lambda)=\lambda\hat{H}(\mu;\lambda^{-1}) \tag{7}$$

and self duality implies $\lambda_c=1$.

Our aim in the following sections is to arrive at this quantum dual Hamiltonian by the other route ADC represented in figure 1.

3. Anisotropic Ising model and its classical dual

For the isotropic Ising model with coupling strength β, the partition function Z can be found to be related to partition function of the dual system as

$$Z(\beta)=\tfrac{1}{2}(\sinh 2\bar{\beta})^{-N}Z(\bar{\beta}) \tag{8}$$

where $\bar{\beta}=-\frac{1}{2}\log\tanh\beta$, the 'dual inverse temperature'. If one naively adopts this procedure for the anisotropic case also, we will get

$$Z(\beta_\tau,\beta)=\tfrac{1}{2}(\sinh 2\bar{\beta}_\tau)^{-N/2}(\sinh 2\bar{\beta})^{-N/2}Z(\bar{\beta}_\tau,\bar{\beta}) \tag{9}$$

$$\bar{\beta}_\tau=-\tfrac{1}{2}\log\tanh\beta_\tau \qquad \bar{\beta}=-\tfrac{1}{2}\log\tanh\beta .$$

To perform the τ-continuum limit for the above in a manner similar to equation (3), we will require $\bar{\beta}$ to go to zero and $\bar{\beta}_\tau$ to go to infinity. This implies that

$$\beta\to\infty \qquad \text{and} \qquad \beta_\tau\to 0 \tag{10}$$

as a consequence of the limiting procedure. Equation (3) and equation (10) are in violent disagreement. Hence the quantum Hamiltonian derived from $Z(\bar{\beta}_\tau;\bar{\beta})$ cannot coincide with the Hamiltonian corresponding to the system at the point D of figure 1 which one naturally expects. If we express $\bar{\beta}=\lambda'\exp(-2\bar{\beta}_\tau)$, λ' cannot be the inverse of λ in equation (3) as is required by equation (7). To overcome this difficulty the obvious solution is to have $\bar{\beta}_\tau$ and $\bar{\beta}$ as couplings along the spatial and time directions respectively in the dual lattice. Having thus exchanged the couplings, we can perform the τ-continuum limit described above:

$$\bar{\beta}_\tau\to\infty, \qquad \bar{\beta}\to 0 \tag{11}$$

with

$$\bar{\beta}_\tau=\delta\exp(-2\bar{\beta}). \tag{12}$$

From equations (9) we see that as $\beta\to 0$ and $\beta_\tau\to\infty$ the following relations

$$\exp(-2\bar{\beta})\to\beta \qquad \text{as} \qquad \beta\to 0 \tag{13}$$

and

$$\bar{\beta}_\tau\to\exp(-2\beta_\tau) \qquad \text{as} \qquad \beta_\tau\to\infty \tag{14}$$

hold which imply that $\exp(-2\beta_\tau)=\delta\beta$.

Comparing (14) and (3) we have $\delta=1/\lambda$ as expected. Hence we conclude that for the plane Ising lattice, duality transformation requires exchange of coupling (see also Rittenberg *et al* 1981).

For a plane anisotropic Ising system we can adopt the method of Savit (1980) in going to the dual system, and the necessary construction lands us in a situation where the two couplings are exchanged. With the notation given in Savit (1980) we have

$$Z(\beta_\tau;\beta)=\sum_{k_\nu}\sum_{k_\mu}\prod_{l_\mu}C_{k_\mu}(\beta_\tau)\prod_{l_\nu}C_{k_\nu}(\beta)\prod_i 2\delta\Big(\sum_i k_\mu+\sum_i k_\nu\Big) \tag{15}$$

where l_μ and l_ν are the links in the two directions and k_μ and k_ν are indices for the two directions, since here these have to be distinguished due to anisotropy. To each l_μ and l_ν of the original lattice we uniquely associate a pair of spins that lie at the end of the dual lattice link which cuts the given link of the original lattice perpendicularly. Thus the coupling strength gets exchanged and we have (see also Syozi 1972)

$$Z(\beta_\tau;\beta)=\tfrac{1}{2}(\sinh 2\bar{\beta}_\tau)^{-N/2}(\sinh 2\bar{\beta})^{-N/2}Z(\bar{\beta};\bar{\beta}_\tau) \tag{16}$$

Turban (1982) and Alcaraz and Koberle (1981) and Rittenberg *et al* (1981) have studied the τ-continuum limit for two-dimensional spin models with Z_N symmetry and have constructed quantum dual along the route ABC of figure 1. Study of the Z_N symmetric models through the route ADC detailed in this paper is in progress†.

4. Transfer matrix method

In this section we wish to point out that in his classic paper Onsager (1944) has constructed the transfer matrix for the $d=2$ Ising lattice from which we can obtain the continuum limit in a direct fashion. The transfer matrix (see Huang 1963 pp. 353–5 equations 17.36–17.40) is given as

$$T=[2\sinh 2\beta]^{N/2}V_2V_1$$

where

$$V_1=\prod_{\alpha=1}^{N}\exp(\bar{\beta}_\tau X_\alpha) \qquad \text{and} \qquad V_2=\prod_{\alpha=1}^{N}\exp(\beta Z_\alpha Z_{\alpha+1})$$

and

$$X_\alpha=I\times I\times I\ldots I\times \underset{\alpha\text{th place}}{\underset{\downarrow}{\sigma_1}}\times I\ldots\times I$$

$$Z_\alpha=I\times I\times I\ldots I\times \underset{\alpha\text{th place}}{\underset{\downarrow}{\sigma_3}}\times I\ldots\times I \tag{17}$$

and $\bar{\beta}_\tau$ and β are the coupling strengths discussed in § 2. Going to the τ-continuum limit discussed earlier, we are in the regime $\bar{\beta}_\tau\to 0$ and $\beta\to 0$ and therefore we can expand the exponential up to first order and obtain the quantum Hamiltonian H obtained earlier as in Savit (1980) in a direct fashion. We can also write the transfer matrix of the dual classical system directly and see the necessity for exchanging the couplings. In Stephen and Mittag (1972 equations (1.2) and (1.6)) this fact is transparently obtained by the dual transformation.

† Duality transformations for many component spin models can also be treated similarly to the way we have carried out Savit's procedure and should lead to the exchange of couplings. However in effecting the classical duality transformations for the vector Potts model, equation (3.8) of Savit (1980) implies that in the expansion of log $C_k(\beta)$ one can separate the dependence of β and that of k in each term of the expansion. It should be pointed out that this is possible only in the limit when β tends to ∞. In view of this it is difficult to obtain the dual form as in equation (3.11) of Savit (1980) for finite β.

5. Remarks

Finally we want to point out that this feature relating to exchange of coupling strength in performing the dual transformation leads us to the study of following questions. It is possible that the analysis of dual transformations in anisotropic lattices with randomness in only one direction may throw up some interesting physical features. The formulation of τ-continuum limit in triangular, honeycomb and other types of lattices should be non-trivial problems. Perhaps the techniques of Baxter and Enting (1978) converting such lattices into square lattices may prove useful in this context. Generalisation of this analysis to lattice spin systems with the presence of additional interactions like second-nearest neighbour etc poses great difficulties conceptually and otherwise. Implications of this 'exchange feature' in duality transformations in higher-dimensional systems deserve very careful analysis (see also Amit 1982).

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